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The Fourier-Sato Transformation of Pure Sheaves

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§0. INTRODUCTION.

Kashiwara-Schapira introduced the notion of pure sheaves in [**K-S**] in order to calculate the shifts which appear when contact transformations are applied to sheaves. The purity of a sheaf describes the obstruction for the prolongment of its sections across critical points of Morse functions and played an important role in studying $\mathbf{R}$-constructible sheaves and, in particular, their index theorems (see M. Kashiwara [**K**] and P. Schapira and N. Tose [**S-T**]). Under the assumption of purity, the obstruction is expressed as cohomology groups, which can be calculated with two microlocal data, the Lagrangian variety associated to the Morse function and that of the micro-support of the sheaf. Then we use the inertia index of three Lagrangian planes. Kashiwara-Schapira studied the functorial properties of pure sheaves by several fundamental operators in [**K-S**]. The Fourier-Sato transformation is a geometric counterpart of Fourier transformation, which is introduced by Sato et al. [**S-K-K**] when they constructed the sheaf of microfunctions. The Fourier-Sato transformation of a conic sheaf on a real vector bundle E is a conic object on the dual bundle E^*. In the category of F_q, this transformation is closely related with the Gauss sum, etc.

In this paper, the author calculates the Fourier-Sato transformation of pure sheaves. In §4 we have the result and the proof. In §5 as a corollary of this result, we obtain another proof of the proposition by Kashiwara-Schapira [**K-S2**] which asserts that the Fourier-Sato transform of a perverse sheaf is also perverse. J. L. Brylinski proved analogous propositions in the algebraic category [**B**, corollaire 7.23] and in the category of F_q [**B**, corollaire 9.11]. The important point of the present paper is that we use only techniques purely in the real domain. Thus the proof is independent of the monodromy structure of perverse sheaves.

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§1. Notation and Conventions.

The following notation is taken from **[K-S]**.

1.0. Throughout this paper, let A be a commutative unitary ring with finite global dimension, $Sh(X)$ the abelian category of sheaves of A-modules on a topological space X, $D(X)$ the derived category of $Sh(X)$. We denote by $D^+(X)$ the full subcategory of $D(X)$ consisting of complexes with cohomology bounded from below and by $D^b(X)$ the full subcategory of $D(X)$ consisting of complexes with bounded cohomology. For an object $\mathcal{F}$ of $D(X)$, we denote by $\mathcal{F}[k]$ the object obtained by k-shifts; that is to say $\mathrm{H}^j(\mathcal{F}[k]) = \mathrm{H}^{j+k}(\mathcal{F})$ and $d^n_{\mathcal{F}[k]} = (-1)^k d^{n+k}_{\mathcal{F}}$. Sheaves on X are identified with complexes of $D(X)$ which are concentrated in degree 0. We use usual notation of derived categories and sheaf cohomology. Refer to **[K-S]** for functors, $\underline{\mathrm{Hom}}(\cdot,\cdot)$, $\cdot\otimes\cdot$, f_*, f^{-1}, $f_!$, $f^!$, $\cdot|_Z$, $\cdot\boxtimes\cdot$, $(\cdot)_Z$, $\Gamma_Z(\cdot,\cdot)$, $\Gamma_Z(\cdot)$, orientation sheaf $\underline{or}_X$, relative orientation sheaf $\underline{or}_{Y/X}$ and constant sheaf $\underline{M}_X$.

1.1. TX, T^*X, T_YX, T^*_YX, $\overset{\circ}{T}X$, $\overset{\circ}{T}{}^*X$, $\overset{\circ}{T}_YX$, $\overset{\circ}{T}{}^*_YX$

For a C^∞-real manifold X, TX (resp. T^*X) denotes the tangent (resp. cotangent) bundle to X. If Y is a submanifold of X, T_YX (resp. T_YX) takes for the normal (resp. conormal) budle to Y. $\overset{\circ}{T}X$, $\overset{\circ}{T}{}^*X$, $\overset{\circ}{T}_YX$ and $\overset{\circ}{T}{}^*_YX$ are defined by

$$\overset{\circ}{T}X = TX \setminus T_XX, \quad \overset{\circ}{T}{}^*X = T^*X \setminus T^*_XX$$

$$\overset{\circ}{T}_YX = T_YX \setminus T_YY, \quad \overset{\circ}{T}{}^*_YX = T^*_YX \setminus T^*_YY.$$

1.2. ϖ_f, ρ_f

For a C^∞-map between C^∞-real manifolds $f : Y \to X$, ϖ_f and ρ_f are defined by

$$\begin{array}{ccc} & Y \underset{X}{\times} T^*X & \\ {}^{\rho_f}\swarrow & & \searrow^{\varpi_f} \\ T^*Y & & T^*X. \end{array}$$

1.3. a

For a vector bundle $E \to Z$, a is an antipodal map in E. If G is a subset of E, G^a is the image of G by this map.

1.4. Micro-support

We recall the definition of micro-support for sheaves.

Definition ([**K-S**]). *Let X be a C^∞-real manifold and $\mathcal{F}$ an object of $D^+(X)$. Then the micro-support of $\mathcal{F}$, denoted* $\mathrm{SS}(\mathcal{F})$, *is a subset of T^*X defined as follows.*

Let U be an open subset of T^*X. Then

$$U \cap \mathrm{SS}(\mathcal{F}) = \emptyset$$

$$\Longleftrightarrow \begin{cases} \text{for any real } C^\infty\text{-function } \phi \text{ on } X, \\ (x_1; d\phi(x_1)) \in U \quad \text{implies} \quad (\mathrm{R}\Gamma_{\{\phi(x) \geqq \phi(x_1)\}}(\mathcal{F}))_{x_1} = 0. \end{cases}$$

1.5. $D^+(X;\Omega)$

Consider the same situation as above. Let Ω be a subset of T^*X. Then $S(\Omega)$ is the set of arrows in $D^+(X)$, given as follows. $f : \mathcal{F} \longrightarrow \mathcal{G}$ belongs to $S(\Omega)$ if there exists a distinguished triangle

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "+1"'] & \\ \mathcal{F} \arrow[rr, "f"] & & \mathcal{G} \arrow[ul] \end{tikzcd}$$

which satisfies

$$\mathrm{SS}(\mathcal{H}) \cap \Omega = \emptyset.$$

The set $S(\Omega)$ is a multiplicative system of $D^+(X)$. Then $D^+(X;\Omega)$ is defined as the localization of $D^+(X)$ with respect to $S(\Omega)$.

§2. The Fourier-Sato Transformation.

2.1. We recall the definition of the Fourier-Sato transformaion from [**K-S**]. The notion of Fourier-Sato transformation is due to Sato-Kashiwara-Kawai ([**S-K-K**]) although they defined it for sphere bundles. Let $E \xrightarrow{\tau} Z$ be a real vector budle with finite fibre dimension over a locally compact topological space Z and $D^+_{conic}(E)$ be the full subcategory of $D^+(E)$ consisting of complexes whose cohomology groups are locally constant on any half-line of E. Let $E^* \xrightarrow{\pi} Z$ be a dual vector bundle of E. Set

$$D^+ = \{(x,y) \in E \underset{Z}{\times} E^* | < x, y > \geqq 0\},$$
$$D^- = \{(x,y) \in E \underset{Z}{\times} E^* | < x, y > \leqq 0\}.$$

Consider the diagram

$$\begin{tikzcd} & E \underset{Z}{\times} E^* & \\ E^* & D^{\pm} \arrow[u, hook] & E. \arrow[from=1-2, to=2-1, "p_2"'] \arrow[from=1-2, to=2-3, "p_1"] \end{tikzcd}$$

For an object $\mathcal{F}$ of $D^+_{conic}(E)$, we define the Fourier-Sato transform $\mathcal{F}^\wedge$ of $\mathcal{F}$ by

$$\mathcal{F}^\wedge = \mathrm{R}\,p_{2*}\,\mathrm{R}\Gamma_{D^+}(p_1^{-1}\mathcal{F}) = \mathrm{R}\,p_{2!}(p_1^{-1}\mathcal{F})_{D^-}.$$

2.2. micro-support of $\mathcal{F}^\wedge$

Let (z) be a coordinate system of Z, (z,x) that of E and $(z,x;\zeta,\xi)$ the associated coordinate system of T^*E. Let (z,y) be a coordinate system of E^* and $(z,y;\zeta,\eta)$ the associated coordinate system of T^*E^* for which the canonical pairing between E and E^* is given by

$$< x, y >= \sum_i x_i y_i$$

and for which the canonical 1-forms of T^*E and T^*E^* are given respectively by

$$\omega_E =< \zeta, dz > + < \xi, dx >$$

and

$$\omega_{E^*} =< \zeta, dz > + < \eta, dy > .$$

Then the canonical isomorphism

$$\Phi_E : T^*E \xrightarrow{\cong} T^*E^*$$

is defined by

$$(z, x; \zeta, \xi) \longmapsto (z, \xi; \zeta, -x).$$

Under the above situation, we have

THEOREM 2.2.1([**K-S**, THEOREM 5.1.4]).

$$\mathrm{SS}(\mathcal{F}^\wedge) = \Phi_E(\mathrm{SS}(\mathcal{F})).$$

2.3. Another proposition from [**K-S**].

PROPOSITION 2.3.1. *Let Y be a real C^∞-manifold and E a real vector space with finite dimension. Let G be a closed convex cone (not necessarily proper) in E with $0 \in G$. Set $X = Y \times E$ and $X_G = Y \times E_G$. Here E_G is the space E endowed with G-topology (see [**K-S**] for definition). Let ϕ be the natural continuous map*

$$\phi : X \longrightarrow X_G.$$

Then following claims hold.

(a) For $\mathcal{F} \in \mathrm{Ob}(D^+(X))$, $\mathrm{SS}(\mathcal{F})$ is contained in $T^*Y \times (E \times G^{\circ a})$ if and only if the morphism $\phi^{-1} \mathrm{R}\phi_* \mathcal{F} \to \mathcal{F}$ is an isomorphism.

(b) For $\mathcal{F} \in \mathrm{Ob}(D^+(X))$, we have

$$\phi^{-1} \mathrm{R}\phi_* \mathcal{F} \cong \mathcal{F} \quad in \quad D^+(X; T^*Y \times (E \times \mathrm{Int}\, G^{\circ a})).$$

§3. PURE SHEAVES.

We recall the definition of pure sheaves from [**K-S**].

3.1. Inertia index $\tau(\lambda_1, \lambda_2, \lambda_3)$

Let (E, σ) be a real symplectic finite dimensional vector space; i.e. σ is a non-degenerate skew symmetric bilinear form on the finite dimensional $\mathbf{R}$-vector space E. Let ρ be a linear subspace of E. Set

$$\rho^\perp = \{x \in E | \sigma(x, y) = 0 \quad \text{for} \quad \forall y \in \rho\}.$$

Then ρ is called Lagrangian if $\rho^\perp = \rho$, involutive if $\rho^\perp \subset \rho$ and isotropic if $\rho^\perp \supset \rho$.

DEFINITION 3.1.1([**K-S**]). *Let λ_1, λ_2, λ_3 be Lagrangian planes of E. Here the quadratic form Q on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ is defined by*

$$Q(x_1, x_2, x_3) = \sigma(x_1, x_2) + \sigma(x_2, x_3) + \sigma(x_3, x_1),$$

for $(x_1, x_2, x_3) \in \lambda_1 \oplus \lambda_2 \oplus \lambda_3$. Then the index $\tau_E(\lambda_1, \lambda_2, \lambda_3)$ is defined as the signature of Q, that is the difference of the number of positive eigenvalues and that of negative eigenvalues of Q.

3.2. Properties of the inertia index

In the following part of this paper, we write τ for τ_E if there is no fear of confusion. Let ρ be an isotropic subspace of E and λ a subset of E. Then λ^ρ is defined by

$$\lambda^\rho = ((\lambda \cap \rho^\perp) + \rho)/\rho.$$

PROPOSITION 3.2.1([**K-S**, PROPOSITION 7.1.2]). *Let λ_i be Lagrangian planes of E. Then we have following statements.*

(i) For all $s \in \mathfrak{S}_3$,

$$\tau(\lambda_1, \lambda_2, \lambda_3) = \mathrm{sgn}(s)\tau(\lambda_{s(1)}, \lambda_{s(2)}, \lambda_{s(3)})$$

holds.

(ii) If ρ is a subspace and satisfies

$$\rho \subset (\lambda_1 \cap \lambda_2) + (\lambda_2 \cap \lambda_3) + (\lambda_3 \cap \lambda_1),$$

then we have

$$\tau_E(\lambda_1, \lambda_2, \lambda_3) = \tau_{E^\rho}(\lambda_1^\rho, \lambda_2^\rho, \lambda_3^\rho).$$

In particular if

$$\lambda_1 \cap (\lambda_2 + \lambda_2) \subset (\lambda_1 \cap \lambda_2) + (\lambda_1 \cap \lambda_3)$$

holds, we have

$$\tau(\lambda_1, \lambda_2, \lambda_3) = 0.$$

3.3. Definition of pure sheaves

Let X be a C^∞-real manifold, π the projection $T^*X \to X$, Λ a Lagrangian submanifold of T^*X, ϕ a real function on X and $Y_\phi = \{(x, d\phi(x)); x \in X\}$. For any point p in T^*X, T_pT^*X has a canonical structure of symplectic vector space. Then three Lagrangian planes in T_pT^*X are defined by

$$\lambda_0(p) = T_p(\pi^{-1}\pi(p)),$$
$$\lambda_\Lambda(p) = T_p\Lambda$$

and

$$\lambda_\phi(p) = T_pY_\phi.$$

DEFINITION 3.3.1. *Under the above situation, we say that ϕ is transversal to Λ at p if $\phi(\pi(p)) = 0$ and if Y_ϕ and Λ intersect transversally at p.*

LEMMA 3.3.2. *Let Λ be a Lagrangian submanifold of T^*X, p a point of Λ and $\mathcal{F}$ an object of $D^+(X)$. Assume that in a neighborhood of p, $\mathrm{SS}(\mathcal{F}) \subset \Lambda$ holds. Let ϕ be a real function on X and transversal to Λ at p. Let j be a number which satisfies*

$$j \equiv \frac{1}{2}(\dim X + \dim(\lambda_0(p) \cap \lambda_\Lambda(p))) \mod \mathbb{Z}.$$

Then the cohomology group

$$\mathrm{H}^{j+\frac{1}{2}\tau_\phi(p)}_{\{x|\phi(x)\geq 0\}}(\mathcal{F})_{\pi(p)}$$

does not depend on ϕ where

$$\tau_\phi(p) = \tau(\lambda_0(p), \lambda_\Lambda(p), \lambda_\phi(p)).$$

After these preparations, we can define pure sheaves.

DEFINITION 3.3.3([**K-S**]). *Let Λ be a Lagrangian submanifold of T^*X, $p \in \Lambda$, and $\mathcal{F} \in \mathrm{Ob}(D^+(X))$. We assume $\mathrm{SS}(\mathcal{F}) \subset \Lambda$ in a neighborhood of p. If we have, for a real function ϕ transversal to Λ at p and A-module M,*

$$\mathrm{H}^j_{\{x|\phi(x)\geqq 0\}}(\mathcal{F})_{\pi(p)} = \begin{cases} M, & \text{for } j = -d + \frac{1}{2}\dim X + \frac{1}{2}\tau_\phi(p); \\ 0, & \text{otherwise} \end{cases}$$

with $\tau_\phi(p) = \tau(\lambda_0(p), \lambda_\Lambda(p), \lambda_\phi(p))$, then we say that $\mathcal{F}$ is pure of type M with shift d along Λ at p.

3.4. Properties of pure sheaves

We recall properties of pure sheaves from [**K-S**].

PROPOSITION 3.4.1 ([**K-S**, PROPOSITION 7.2.8, 7.2.9]).

(i) Let Λ be a Lagrangian submanifold of $\overset{\circ}{T}{}^*X$, p a point of Λ and $\mathcal{F}$ an object of $D^b(X)$. Assume that $\mathcal{F}$ is pure of type M with shift d along Λ at p and that $\mathrm{Ext}^j(M, A) = 0$ $(j \neq 0)$. Then $\mathrm{R}\,\underline{\mathrm{Hom}}(\mathcal{F}, \underline{A}_X)$ is pure of type $\mathrm{Hom}(M, A)$ with shift $-d$ along Λ^a at p^a.

(ii) Let Λ_j be a Lagrangian submanifold of T^*X_j, p_j a point of Λ_j and $\mathcal{F}_j$ an object of $D^+(X_j)$. Assume that $\mathcal{F}_j$ is pure of type M_j with shift d_j along Λ_j at p_j $(j = 1, 2)$. Let q_j be the j-th projection on $X_1 \times X_2$.

(a) If $\mathrm{Tor}_j(M_1, M_2) = 0$ for $\forall j \neq 0$, then $q_1^{-1}\mathcal{F}_1 \overset{\mathrm{L}}{\otimes} q_2^{-1}\mathcal{F}_2$ is pure of type $M_1 \otimes M_2$ with shift $d_1 + d_2$ along $\Lambda_1 \times \Lambda_2$ at (p_1, p_2).

(b) If $\mathrm{Ext}^j(M_1, M_2) = 0$ for $\forall j \neq 0$, then $\mathrm{R}\,\underline{\mathrm{Hom}}(q_1^{-1}\mathcal{F}_1, q_2^{-1}\mathcal{F}_2)$ is pure of type $\mathrm{Hom}(M_1, M_2)$ with shift $d_2 - d_1$ along $\Lambda_1^a \times \Lambda_2$ at (p_1^a, p_2).

Let $f : Y \to X$ be a C^∞-map between C^∞-manifolds.

THEOREM 3.4.2 ([**K-S**, THEOREM 7.3.1]). *Let Λ be a Lagrangian submanifold of T^*Y, p a point of $Y \underset{X}{\times} T^*X$ and $\mathcal{G}$ an object of $D^+(Y)$. Assume:*

(i) f is proper over $\mathrm{supp}(\mathcal{G})$,

(ii) ρ_f is transversal to Λ at p and $\varpi_f\rho_f^{-1}(\Lambda)$ is isomorphic to a submanifold Λ_0 of T^*X,

(iii) $\rho_f^{-1}(\mathrm{SS}(\mathcal{G})) \cap \varpi_f^{-1}\varpi_f(p) \subset \{p\}$,

(iv) $\mathcal{G}$ is pure of type M with shift d along Λ at $\rho_f(p)$.

Then Λ_0 is a Lagrangian submanifold and $\mathrm{R}\, f_*(\mathcal{G})$ is pure of type M with shift d' along Λ_0 at $\varpi_f(p)$ where

$$d' - d = \frac{1}{2}(\dim X - \dim Y) - \frac{1}{2}\tau(\lambda_0(\rho_f(p)), \lambda_\Lambda(\rho_f(p)), \rho_f\varpi_f^{-1}(\lambda_0(\varpi_f(p)))).$$

THEOREM 3.4.3 ([**K-S**, THEOREM 7.3.3]). *f, X, Y are the same as those of Theorem 3.4.2. Let Λ be a Lagrangian submanifold of T^*X, p a point of $Y \underset{X}{\times} \Lambda$ and $\mathcal{F}$ an object of $D^+(X)$. Assume:*

(i) f is non-characteristic for $\mathcal{F}$,

(ii) ϖ_f is transversal to Λ at p and $\rho_f \varpi_f^{-1}(\Lambda)$ is isomorphic to a submanifold Λ_0 of T^*Y,

(iii) $\varpi_f^{-1}(\mathrm{SS}(\mathcal{F})) \cap \rho_f^{-1}\rho_f(p) \subset \{p\}$,

(iv) $\mathcal{F}$ is pure of type M with shift d along Λ at $\varpi_f(p)$.

Then Λ_0 is a Lagrangian submanifold and $f^{-1}(\mathcal{F})$ is pure of type M with shift d along Λ_0 at $\rho_f(p)$.

THEOREM 3.4.4 ([**K-S**, COROLLARY 7.3.4]). *Let X and Y be C^∞-real manifolds, q_1 and q_2 the projections from $X \times Y$ to X and Y respectively and p_1 and p_2 the projections from $T^*(X \times Y) = T^*X \times T^*Y$ to T^*X and T^*Y respectively. Set $p_j^a = p_j \circ a$ $(j = 1, 2)$. Let Λ be a Lagrangian submanifold of $T^*(X \times Y)$, Λ_Y a Lagrangian submanifold of T^*Y and p a point of Λ. Set $p_Y = p_2(p)$ and $p_X = p_1^a(p)$. Let $\mathcal{K}$ be an object of $D^b(X \times Y)$ and $\mathcal{F}$ an object of $D^+(Y)$. Assume:*

(i) $p_2|_\Lambda$ is transversal to Λ_Y at p and $p_2^{-1}(\Lambda_Y) \cap \Lambda$ is isomorphic to a submanifold Λ_X of T^*X by p_1^a,

(ii) $\mathcal{K}$ is pure of type M with shift d along Λ at p,

(iii) $\mathcal{F}$ is pure of type N with shift d' along Λ_Y at p_Y,

(iv) q_1 is proper over $\mathrm{supp}(\mathcal{K}) \cap q_2^{-1}(\mathrm{supp}(\mathcal{F}))$,

(v) $(p_1^a)^{-1}(p_x) \cap \mathrm{SS}(\mathcal{K}) \subset \{p\}$,

(vi) $(\mathrm{SS}(\mathcal{K}) \underset{T^*Y}{\times} \mathrm{SS}(\mathcal{F})) \cap (T_X^*X \times T^*Y) \subset T_X^*X \times T_Y^*Y$ holds in a neighborhood of $\pi_X(p_X)$.

(vii) $\mathrm{Ext}^j(M, N) = 0$ for $\forall j \neq 0$.

Then $\mathrm{R}q_{1*}\,\mathrm{R}\underline{\mathrm{Hom}}(\mathcal{K}, q_2^{-1}\mathcal{F})$ is pure of type $\mathrm{Hom}(M, N)$ with shift d'' along Λ_X at p_X where

$$d'' = d' - d - \frac{1}{2}\dim Y + \frac{1}{2}\tau$$

and

$$\begin{aligned}\tau &= \tau(\lambda_0(p), \lambda_\Lambda(p), \lambda_0(p_X^a) \times \lambda_{\Lambda_Y}(p_Y)) \\ &= \tau(\lambda_0(p_Y), p_2(\lambda_\Lambda(p) \cap (p_1^a)^{-1}(\lambda_0(p_X))), \lambda_{\Lambda_Y}(p_Y)).\end{aligned}$$

This proposition describes the contact transformation of pure sheaves.

3.5. Microlocal uniqueness of pure sheaves.

The following fact is important.([**K-S**]) Let $\mathcal{F}$ and $\mathcal{G}$ be objects of $D^+(X)$. Assume $\mathcal{F}$ is pure of type M with shift d along Λ at p. Then $\mathcal{G}$ is pure of type M with shift d along Λ at p if and only if

$$\mathcal{F} \cong \mathcal{G} \quad \text{in} \quad D^+(X; \{p\})$$

holds.

§4. The Fourier-Sato Transformation of Pure Sheaves.

4.1. The Main Theorem.

THEOREM 4.1.1. *Let $E \to Z$ be an $\mathbf{R}$-vector bundle with finite fibre dimension n over a locally compact topological space Z and Λ a Lagrangian submanifold of T^*E. Let $\mathcal{F} \in \mathrm{Ob}(D^+_{conic}(E))$ and $p \in \Lambda$. Assume $\mathcal{F}$ is pure of type M with shift d along Λ at p. Then the Fourier-Sato transform $\mathcal{F}^\wedge$ of $\mathcal{F}$ is pure of type M with shift d' along Λ^* at p^* where*

$$p^* = \Phi_E(p),$$
$$\Lambda^* = \Phi_E(\Lambda),$$

and

$$d' = d - \frac{n}{2} + \frac{1}{2}\tau(\lambda_0(p), \Phi_E^{-1}(\lambda_0(p^*)), \lambda_\Lambda(p)).$$

PROOF: Identify Z with the zero sections of E and E^*. Set

$$\overset{\circ}{E} = E \setminus Z, \quad \overset{\circ}{E^*} = E^* \setminus Z,$$
$$S = \overset{\circ}{E}/\mathbf{R}^+, \quad S^* = \overset{\circ}{E^*}/\mathbf{R}^+,$$
$$D_E = \{(x,y) \in E \underset{Z}{\times} E^* \mid <x,y> \geqq 0\},$$
$$D_{\overset{\circ}{E}} = \{(x,y) \in \overset{\circ}{E} \underset{Z}{\times} \overset{\circ}{E^*} \mid <x,y> \geqq 0\},$$
$$D_S = \{(x,y) \in S \underset{Z}{\times} S^* \mid <x,y> \geqq 0\},$$
$$D_{E\overset{\circ}{E}} = D_E \underset{E^* \underset{Z}{\times} E}{\times} (E^* \underset{Z}{\times} \overset{\circ}{E}),$$
$$D_{S\overset{\circ}{E}} = D_S \underset{S \underset{Z}{\times} S^*}{\times} (S^* \underset{Z}{\times} \overset{\circ}{E}).$$

First we give three lemmas.

LEMMA 4.1.2. *Consider the diagram*

$$
\begin{tikzcd}
 & & \mathring{E}^* \underset{Z}{\times} \mathring{E} \arrow[dll, "p_{12}"'] \arrow[dr, "p_{11}"] \arrow[dd, "s_1"] & & D_{\mathring{E}} \arrow[ll, hook', "i_1"'] \arrow[dd, "s_3"] \\
\mathring{E}^* \arrow[dd, "f_2"] & & & \mathring{E} \arrow[dd, "f_1"] & \\
 & & S^* \underset{Z}{\times} \mathring{E} \arrow[ur, "p_{21}"] \arrow[dll, "p_{22}"'] \arrow[dd, "s_2"] & & D_{S\mathring{E}} \arrow[ll, hook', "i_2"'] \arrow[dd, "s_4"] \\
S^* & & & S & \\
 & & S^* \underset{Z}{\times} S \arrow[ull, "p_{32}"] \arrow[ur, "p_{31}"'] & & D_S. \arrow[ll, hook', "i_3"']
\end{tikzcd}
$$

Then for an arbitrary object $\mathcal{F}$ of $D^+_{conic}(\mathring{E})$, we have

$$\mathcal{F}^{\wedge\circ} = f_2^{-1}((\mathrm{R} f_{1*}\mathcal{F})^{\wedge S}).$$

Here for $\mathcal{F} \in \mathrm{Ob}(D^+_{conic}(\mathring{E}))$ and $\mathcal{G} \in \mathrm{Ob}(D^+(S))$, $\mathcal{F}^{\wedge\circ}$ and $\mathcal{G}^{\wedge S}$ are defined by

$$\mathcal{F}^{\wedge\circ} = \mathrm{R} p_{12*}\, \mathrm{R}\Gamma_{D_{\mathring{E}}} p_{11}{}^{-1}\mathcal{F}$$
$$\mathcal{G}^{\wedge S} = \mathrm{R} p_{32*}\, \mathrm{R}\Gamma_{D_S} p_{31}{}^{-1}\mathcal{G}.$$

PROOF OF LEMMA 4.1.2: Since $\mathrm{R}\Gamma_{D_S}(\cdot) = \mathrm{R} i_{3*} i_3^!(\cdot)$, we have

(4.1.2.1)
$$
\begin{aligned}
f_2^{-1}((\mathrm{R} f_{1*}\mathcal{F})^{\wedge S}) &= f_2^{-1}\, \mathrm{R} p_{32*}\, \mathrm{R}\Gamma_{D_S} p_{31}^{-1}\, \mathrm{R} f_{1*}\mathcal{F} \\
&= f_2^{-1}\, \mathrm{R} p_{32*}\, \mathrm{R} i_{3*} i_3^! p_{31}^{-1}\, \mathrm{R} f_{1*}\mathcal{F}.
\end{aligned}
$$

From the fact that s_2 is a topological submersion of codimension 1, we get, by Poincaré-Verdier duality theorem,

(4.1.2.2) $$p_{31}^{-1}\, \mathrm{R} f_{1*}\mathcal{F} \otimes \underline{or}_{(S^* \underset{Z}{\times} \mathring{E})/(S^* \underset{Z}{\times} S)}[1] = \mathrm{R} s_{2*} s_1^! p_{31}^{-1}\, \mathrm{R} f_{1*}\mathcal{F}.$$

Now we remark that the following part of the diagram in Lemma 4.1.2 is a Cartesian diagram.

$$\begin{tikzcd} S^* \times_Z \overset{\circ}{E} \arrow[d, "s_2"] & D_{S\overset{\circ}{E}} \arrow[l, "i_2"'] \arrow[d, "s_4"] \\ S^* \times_Z S & D_S \arrow[l, "i_3"'] \end{tikzcd}$$

From (4.1.2.1),(4.1.2.2) and this fact, we deduce

(4.1.2.3)
$$\begin{aligned} &f_2^{-1}((\mathrm{R}\, f_{1*}\mathcal{F})^{\wedge S}) \otimes \underline{or}_{(S^* \times_Z \overset{\circ}{E})/(S^* \times_Z S)}[1] \\ =&f_2^{-1}\,\mathrm{R}\, p_{32*}\,\mathrm{R}\, i_{3*}i_3^!\,\mathrm{R}\, s_{2*}s_2^! p_{31}^{-1}\,\mathrm{R}\, f_{1*}\mathcal{F} \\ =&f_2^{-1}\,\mathrm{R}\, p_{32*}\,\mathrm{R}\, i_{3*}\,\mathrm{R}\, s_{4*}i_2^! s_2^! p_{31}^{-1}\,\mathrm{R}\, f_{1*}\mathcal{F} \\ =&f_2^{-1}\,\mathrm{R}\, p_{32*}\,\mathrm{R}\, s_{2*}\,\mathrm{R}\, i_{2*}i_2^! s_2^! p_{31}^{-1}\,\mathrm{R}\, f_{1*}\mathcal{F} \\ =&f_2^{-1}\,\mathrm{R}\, p_{22*}\,\mathrm{R}\, i_{2*}i_2^! s_2^! p_{31}^{-1}\,\mathrm{R}\, f_{1*}\mathcal{F}. \end{aligned}$$

Since p_{31} is a topological submersion of codimension $(n-1)$,

(4.1.2.4)
$$p_{31}^{-1}\,\mathrm{R}\, f_{1*}\mathcal{F} \otimes \underline{or}_{(S^* \times_Z S)/S}[n-1] = p_{31}^!\,\mathrm{R}\, f_{1*}\mathcal{F}\ .$$

Remarking that s_2 is a topological submersion and that the diagram

$$\begin{tikzcd} S^* \times_Z \overset{\circ}{E} \arrow[r, "p_{21}"] \arrow[d, "s_2"] & \overset{\circ}{E} \arrow[d, "f_1"] \\ S^* \times_Z S \arrow[r, "p_{31}"] & S \end{tikzcd}$$

is a Cartesian diagram, we have, by (4.1.2.3) and (4.1.2.4),

(4.1.2.5)
$$\begin{aligned} &f_2^{-1}((\mathrm{R}\, f_{1*}\mathcal{F})^{\wedge S}) \otimes \underline{or}_{(S^* \times_Z \overset{\circ}{E})/S}[n] \\ =&f_2^{-1}\,\mathrm{R}\, p_{22*}\,\mathrm{R}\, i_{2*}i_2^! s_2^! p_{31}^!\,\mathrm{R}\, f_{1*}\mathcal{F} \\ =&f_2^{-1}\,\mathrm{R}\, p_{22*}\,\mathrm{R}\, i_{2*}i_2^! s_2^!\,\mathrm{R}\, s_{2*}p_{21}^!\mathcal{F} \\ =&f_2^{-1}\,\mathrm{R}\, p_{22*}\,\mathrm{R}\, i_{2*}i_2^! p_{21}^!\mathcal{F} \otimes \underline{or}_{(S^* \times_Z \overset{\circ}{E})/(S^* \times_Z S)}[1]\ . \end{aligned}$$

Now s_1 is a topological submersion of codimension 1 and the diagram

$$
\begin{array}{ccc}
\overset{\circ}{E} \underset{Z}{\times} \overset{\circ}{E}^* & \xleftarrow{\ i_1\ } & D_{\overset{\circ}{E}} \\
{\scriptstyle s_1}\big\downarrow & & \big\downarrow{\scriptstyle s_3} \\
S^* \underset{Z}{\times} \overset{\circ}{E} & \xleftarrow{\ i_2\ } & D_{S\overset{\circ}{E}}
\end{array}
$$

is Cartesian. Thus we have from (4.1.2.5),

$$
\begin{aligned}
& f_2^{-1}((\mathrm{R} f_{1*}\mathcal{F})^{\wedge S}) \otimes \underline{or}_{(\overset{\circ}{E^*} \underset{Z}{\times} \overset{\circ}{E})/S}[n] \\
=& f_2^{-1}\,\mathrm{R} p_{22*}\,\mathrm{R} i_{2*} i_2^!\,\mathrm{R} s_{1*} s_1^! p_{21}^! \mathcal{F} \\
=& f_2^{-1}\,\mathrm{R} p_{22*}\,\mathrm{R} i_{2*}\,\mathrm{R} s_{3*} i_1^! s_1^! p_{21}^! \mathcal{F} \\
=& f_2^{-1}\,\mathrm{R} p_{22*}\,\mathrm{R} s_{1*}\,\mathrm{R} i_{1*} i_1^! s_1^! p_{21}^! \mathcal{F} \\
=& f_2^{-1}\,\mathrm{R} f_{2*}\,\mathrm{R} p_{12*}\,\mathrm{R} i_{1*} i_1^! s_1^! p_{21}^! \mathcal{F} \\
=& \mathrm{R} p_{12*}\,\mathrm{R} i_{1*} i_1^! p_{11}^! \mathcal{F} = (*).
\end{aligned}
$$

Since p_{11} is a topological submersion of codimension n, we have, moreover,

$$
\begin{aligned}
(*) =& \mathrm{R} p_{12*}\,\mathrm{R} i_{1*} i_1^! p_{11}^{-1} \mathcal{F} \otimes \underline{or}_{(\overset{\circ}{E^*} \underset{Z}{\times} \overset{\circ}{E})/\overset{\circ}{E}}[n] \\
=& \mathrm{R} p_{12*}\,\mathrm{R}\Gamma_{D_{\overset{\circ}{E}}} p_{11}^{-1} \mathcal{F} \otimes \underline{or}_{(\overset{\circ}{E^*} \underset{Z}{\times} \overset{\circ}{E})/\overset{\circ}{E}}[n] \\
=& \mathcal{F}^{\wedge\circ} \otimes \underline{or}_{(\overset{\circ}{E^*} \underset{Z}{\times} \overset{\circ}{E})/\overset{\circ}{E}}[n]\ .
\end{aligned}
$$

Then we have

$$
f_2^{-1}((\mathrm{R} f_{1*}\mathcal{F})^{\wedge S}) = \mathcal{F}^{\wedge\circ}.
$$

∎

LEMMA 4.1.3. *Consider the diagram*

$$\begin{tikzcd}
 & & \mathring{E}^* \underset{Z}{\times} \mathring{E} \arrow[dll, "p_{12}"'] \arrow[dr, "p_{11}"] \arrow[dd, hook, "i_1"] & & D_{\mathring{E}} \arrow[ll, hook', "j_1"'] \arrow[dd, hook, "i_3"] \\
\mathring{E}^* \arrow[dd, hook, "g_2"] & & & \mathring{E} \arrow[dd, hook, "g_1"] & \\
 & & E^* \underset{Z}{\times} \mathring{E} \arrow[dll, "p_{22}"'] \arrow[ur, "p_{21}"] \arrow[dd, hook, "i_2"] & & D_{E\mathring{E}} \arrow[ll, hook', "j_2"'] \arrow[dd, hook, "i_4"] \\
E^* & & & E & \\
 & & E^* \underset{Z}{\times} E \arrow[ull, "p_{32}"] \arrow[ur, "p_{31}"'] & & D_E. \arrow[ll, hook', "j_3"']
\end{tikzcd}$$

Then for an arbitrary object $\mathcal{F}$ of $D^+_{conic}(E)$,

$$\begin{tikzcd}
 & (g_1^{-1}\mathcal{F})^{\wedge\circ} \arrow[dl, "+1"'] & \\
g_2^{-1}q_2^{-1}\,\mathrm{R}q_{1*}\,\mathrm{R}\Gamma_Z\mathcal{F} \arrow[rr] & & g_2^{-1}(\mathcal{F}^{\wedge}) \arrow[ul]
\end{tikzcd}$$

is a distinguished triangle where q_1 is the natural projection $E \xrightarrow{q_1} Z$ and q_2 the natural projection $E^* \xrightarrow{q_2} Z$.

PROOF OF LEMMA 4.1.3: Remark that if i is an open inclusion, $i^!$ coincides with i^{-1}. Now g_1, g_2, i_1, i_2, i_3 and i_4 are open inclusions. Taking into account of the fact that diagrams

$$\begin{tikzcd}
\mathring{E}^* \underset{Z}{\times} \mathring{E} \arrow[d, "i_1"'] & D_{\mathring{E}} \arrow[l, "j_1"'] \arrow[d, "i_3"'] \\
E^* \underset{Z}{\times} \mathring{E} & D_{E\mathring{E}} \arrow[l, "j_2"']
\end{tikzcd}$$

$$\begin{tikzcd}
\overset{\circ}{E}{}^* & \overset{\circ}{E}{}^* \underset{Z}{\times} \overset{\circ}{E} \arrow[l, "p_{12}"'] \\
E^* \arrow[from=u, "g_2"'] & E^* \underset{Z}{\times} \overset{\circ}{E} \arrow[l, "p_{22}"'] \arrow[from=u, "i_1"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
E^* \underset{Z}{\times} \overset{\circ}{E} & D_{E\overset{\circ}{E}} \arrow[l, "j_2"'] \\
E^* \underset{Z}{\times} E \arrow[from=u, "i_2"'] & D_E \arrow[l, "j_3"'] \arrow[from=u, "i_4"']
\end{tikzcd}$$

are Cartesian diagrams, we have by Poincaré-Verdier duality theorem

(4.1.3.1)

$$\begin{aligned}
(g_1^{-1}\mathcal{F})^{\wedge\circ} &= \mathrm{R}p_{12*}\,\mathrm{R}\Gamma_{D_{\overset{\circ}{E}}}p_{11}^{-1}g_1^{-1}\mathcal{F} \\
&= \mathrm{R}p_{12*}\,\mathrm{R}j_{1*}j_1^{!}p_{11}^{-1}g_1^{-1}\mathcal{F} \\
&= \mathrm{R}p_{12*}\,\mathrm{R}j_{1*}j_1^{!}i_1^{-1}p_{21}^{-1}g_1^{-1}\mathcal{F} \\
&= \mathrm{R}p_{12*}\,\mathrm{R}j_{1*}i_3^{!}j_2^{!}p_{21}^{-1}g_1^{-1}\mathcal{F} \\
&= \mathrm{R}p_{12*}i_1^{!}\,\mathrm{R}j_{2*}j_2^{!}p_{21}^{-1}g_1^{-1}\mathcal{F} \\
&= g_2^{!}\,\mathrm{R}p_{22*}\,\mathrm{R}j_{2*}j_2^{!}p_{21}^{-1}g_1^{-1}\mathcal{F} \\
&= g_2^{!}\,\mathrm{R}p_{22*}\,\mathrm{R}j_{2*}j_2^{!}i_2^{-1}p_{31}^{-1}\mathcal{F} \\
&= g_2^{!}\,\mathrm{R}p_{22*}\,\mathrm{R}j_{2*}j_2^{!}i_2^{!}p_{31}^{-1}\mathcal{F} \\
&= g_2^{!}\,\mathrm{R}p_{22*}\,\mathrm{R}j_{2*}i_4^{!}j_3^{!}p_{31}^{-1}\mathcal{F} \\
&= g_2^{-1}\,\mathrm{R}p_{22*}i_2^{!}\,\mathrm{R}j_{3*}j_3^{!}p_{31}^{-1}\mathcal{F} \\
&= g_2^{-1}\,\mathrm{R}p_{22*}i_2^{!}\,\mathrm{R}\Gamma_{D_E}p_{31}^{-1}\mathcal{F} \\
&= g_2^{-1}\,\mathrm{R}p_{32*}\,\mathrm{R}i_{2*}i_2^{!}\,\mathrm{R}\Gamma_{D_E}p_{31}^{-1}\mathcal{F} \\
&= g_2^{-1}\,\mathrm{R}p_{32*}\,\mathrm{R}\Gamma_{E^*\underset{Z}{\times}\overset{\circ}{E}}\,\mathrm{R}\Gamma_{D_E}p_{31}^{-1}\mathcal{F}.
\end{aligned}$$

Apply the functor $g_2^{-1}\,\mathrm{R}p_{32*}\,\mathrm{R}\Gamma_{D_E}$ to the distinguished triangle

$$\begin{tikzcd}
 & \mathrm{R}\Gamma_{E^{*}\underset{Z}{\times}\overset{\circ}{E}}\, p_{31}^{-1}\mathcal{F} \arrow[dl, "+1"'] & \\
\mathrm{R}\Gamma_{E^{*}\underset{Z}{\times}\{0\}}\, p_{31}^{-1}\mathcal{F} \arrow[rr] & & p_{31}^{-1}\mathcal{F}. \arrow[ul]
\end{tikzcd}$$

Then combinig this with (4.1.3.1), we get the distinguished triangle

$$\begin{tikzcd}
 & (g_1^{-1}\mathcal{F})^{\wedge\circ} \arrow[dl, "+1"'] & \\
g_2^{-1}q_2^{-1}\,\mathrm{R}q_{1*}\,\mathrm{R}\Gamma_Z\mathcal{F} \arrow[rr] & & g_2^{-1}(\mathcal{F}^{\wedge}). \arrow[ul]
\end{tikzcd}$$

∎

LEMMA 4.1.4. *Let (z) be a coordinate system of Z, (x) that of S and (y) that of S^*. Define two inclusions $i_1 : S^* \underset{Z}{\times} S \longrightarrow S^* \times S$ and $i_2 : D_S \longrightarrow S^* \times S$ as they embed Z into the diagonal set of $Z \times Z$; i.e.*

$$S^* \underset{Z}{\times} S \ni (z; x, y) \overset{i_1}{\longmapsto} ((z;x),(z;y)) \in S^* \times S$$

$$D_S \ni (z; x, y) \overset{i_2}{\longmapsto} ((z;x),(z;y)) \in S^* \times S.$$

Consider the diagram

$$\begin{tikzcd}
 & S^* \underset{Z}{\times} S \arrow[dl, "\widetilde{p}_2"'] \arrow[d, "i_1"] & D_S \arrow[l, hook', "i_3"'] \arrow[dl, "i_2"'] \arrow[dr, "\widetilde{p}_1"] & \\
S^* & S^* \times S \arrow[l, "p_2"'] \arrow[rr, "p_1"] & & S.
\end{tikzcd}$$

Then for an arbitrary object $\mathcal{F}$ of $D^+(S)$, we have

$$\mathrm{R}\widetilde{p}_{2*}\,\mathrm{R}\Gamma_{D_S}\widetilde{p}_1^{\,-1}\mathcal{F} \otimes \underline{or}_{(S^*\underset{Z}{\times}S)/(S^*\times S)}[-\dim Z]$$

$$= \mathrm{R}p_{2*}\,\mathrm{R}\Gamma_{D_S}p_1^{-1}\mathcal{F}$$

(i.e. $\mathcal{F}^{\wedge S} \otimes \underline{or}_{(S^* \underset{Z}{\times} S)/(S^* \times S)}[-\dim Z] = \mathrm{R}p_{2*}\,\mathrm{R}\Gamma_{D_S} p_1^{-1}\mathcal{F}$).

PROOF OF LEMMA 4.1.4: Remark that $\widetilde{p}_1$ is a topological submersion of codimension $(n-1)$ and that p_1 a topological submersion of codimension $(n-1+\dim Z)$. Then by Poincaré-Verdier duality theorem, we get

$$\begin{aligned}
&\mathrm{R}\widetilde{p}_{2*}\,\mathrm{R}\Gamma_{D_S}\widetilde{p}_1^{\,-1}\mathcal{F} \otimes \underline{or}_{(S^* \underset{Z}{\times} S)/S}[n-1] \\
&= \mathrm{R}\widetilde{p}_{2*}\,\mathrm{R}\Gamma_{D_S}\widetilde{p}_1^{\,!}\mathcal{F} \\
&= \mathrm{R}\widetilde{p}_{2*}\,\mathrm{R}i_{3*}i_3^! i_1^! p_1^! \mathcal{F} \\
&= \mathrm{R}\widetilde{p}_{2*}\,\mathrm{R}i_{3*}i_3^! i_1^! p_1^{-1}\mathcal{F} \otimes \underline{or}_{(S^* \times S)/S}[n-1+\dim Z].
\end{aligned}$$

From this, we deduce

$$\begin{aligned}
&\mathrm{R}\widetilde{p}_{2*}\,\mathrm{R}\Gamma_{D_S}\widetilde{p}_1^{\,-1}\mathcal{F} \otimes \underline{or}_{(S^* \underset{Z}{\times} S)/(S^* \times S)}[-\dim Z] \\
&= \mathrm{R}\widetilde{p}_{2*}\,\mathrm{R}i_{3*}i_3^! i_1^! p_1^{-1}\mathcal{F} \\
&= \mathrm{R}p_{2*}\,\mathrm{R}i_{1*}\,\mathrm{R}i_{3*}i_2^! p_1^{-1}\mathcal{F} \\
&= \mathrm{R}p_{2*}\,\mathrm{R}i_{2*}i_2^! p_1^{-1}\mathcal{F} \\
&= \mathrm{R}p_{2*}\,\mathrm{R}\Gamma_{D_S} p_1^{-1}\mathcal{F}.
\end{aligned}$$

∎

Now we enter into the Proof of Theorem 4.1.1.

1. Proof in the case that $p \in \Lambda \cap \overset{\circ}{T}{}^*\overset{\circ}{E}$.

Since $p^* \notin \mathrm{SS}(q_2^{-1}\,\mathrm{R}q_{1*}\,\mathrm{R}\Gamma_Z\mathcal{F})$ in this case, we have an isomorphism

$$(4.1.1.1) \qquad (g_1^{-1}\mathcal{F})^{\wedge\circ} \cong g_2^{-1}(\mathcal{F}^{\wedge}) \quad \text{in} \quad D^+(\overset{\circ}{E};\{p^*\})$$

by Lemma 4.1.3.

Define

$$\begin{aligned}
p_S &= \varpi_{f_1}\rho_{f_1}^{-1}(p) \\
\Lambda_S &= \varpi_{f_1}\rho_{f_1}^{-1}(\Lambda),
\end{aligned}$$

using the notation in Lemma 4.1.2. It is known by [**K-S**, Proposition 5.1.1], that an object $\mathcal{F}$ of $D^+(E)$ belongs to $\mathrm{Ob}(D^+_{conic}(E))$ if and only if $\mathrm{SS}(F)$ is contained in S_E where S_E is the characteristic variety of the Euler vector field on E; i.e. $S_E = \{(z,x;\zeta,\xi) \in T^*E \mid <x,\xi>=0\}$ where (z,x) is a coordinate system of E with the fibre coordinate system (x) and its dual coordinate system(ζ,ξ). Then we may regard Λ as a Lagrangian submanifold of T^*S, and this coincides with Λ_S. Since p is in $\overset{\circ}{T}{}^*\overset{\circ}{E}$, $\mathcal{F}$ coincides with $g_1^{-1}\mathcal{F}$ in a neighborhood of $\pi(p)$. Consequently $g_1^{-1}\mathcal{F}$ is pure with shift d along Λ at p.

CLAIM 1. *$\mathrm{R}f_{1*}g_1^{-1}\mathcal{F}$ is pure of type M with shift d along Λ_S at p_S.*

PROOF OF CLAIM 1: Let $\mathcal{G} \in \mathrm{Ob}(D^+(S))$, and assume $f_1^{-1}\mathcal{G}$ is pure with shift d along Λ at p. We have an isomorphism

$$f_1^{-1}\mathcal{G} \cong g_1^{-1}\mathcal{F} \quad \text{in} \quad D^+(\mathring{E}; \{p\}).$$

Then by Proposition 2.3.1 we have an isomorphism

$$\mathcal{G} \cong \mathrm{R}f_{1*}g_1^{-1}\mathcal{F} \quad \text{in} \quad D^+(S; \{p_S\}).$$

Remark that pure sheaves are micro-locally unique (see 3.5). Therefore it is enough to show that if $\mathcal{G} \in \mathrm{Ob}(D^+(S))$ is pure with shift d along Λ_S at p_S, $f_1^{-1}\mathcal{G}$ is pure with shift d along Λ at p. Since $\mathring{E} \xrightarrow{f_1} S$ is a projection, f_1 is non-characteristic for $\mathcal{G}$. Now since ϖ_{f_1} is smooth, ϖ_{f_1} is transversal to $\rho_{f_1}^{-1}(\Lambda)$ at the point $p_0 = \rho_{f_1}^{-1}(p)$. Remark that ρ_{f_1} is an injection and $\varpi_{f_1}(p_0) = p_S$. Now we have

$$\rho_{f_1}\varpi_{f_1}^{-1}(\Lambda_S) \cong \Lambda$$

and

$$\varpi_{f_1}^{-1}(\mathrm{SS}(\mathcal{G})) \cap \rho_{f_1}^{-1}\rho_{f_1}(p_0) \subset \{p_0\}.$$

Then it follows from Theorem 3.4.3 (the theorem of inverse image of pure sheaves) that $f_1^{-1}\mathcal{G}$ is pure with shift d along Λ at p. ∎

CLAIM 2. *If $\mathcal{G} \in \mathrm{Ob}(D^+(S))$ is pure of type M with shift d along Λ_S at p_S, then $\mathcal{G}^{\wedge S}$ is pure of type M with shift d' along Λ_S^* at p_S^* where*

$$p_S^* = \Phi_S(p_S)$$
$$\Lambda_S^* = \Phi_S(\Lambda_S)$$

and

$$d' = d - \frac{n}{2} + \frac{1}{2}\tau(\lambda_0(p), \Phi_E^{-1}(\lambda_0(p^*)), \lambda_\Lambda(p)).$$

This claim is essential. The proof depends on the fact that considered on S, Fourier-Sato transformation is a contact transformation. This claim is proved by use of Proposition 3.4.4 (the proposition of contact transformation of pure sheaves).

PROOF OF CLAIM 2: Consider the following diagram and that of Lemma 4.1.4.

$$\begin{array}{ccccc} & & \overset{\circ}{T}{}^*(S^*\times S) & & \\ & \overset{r_2}{\swarrow} & \uparrow i & \overset{r_1}{\searrow} & \\ \overset{\circ}{T}{}^*S^* & & \Lambda_K = \overset{\circ}{T}{}^*_{D_S}(S^*\times S) & & \overset{\circ}{T}{}^*S \end{array}$$

Here D_S is embedded into $S^* \times S$ in the same way as Lemma 4.1.4; i.e. the base space Z of D_S is embedded into the diagonal of $Z \times Z$ in $S^* \times S$. Set

$$\overset{\circ}{D}_S = \{(x,y) \in S^* \underset{Z}{\times} S | < x, y >= 0\}.$$

Here (x) is the fibre coordinate system of S and (y) its dual coordinate system. Remark that

$$\overset{\circ}{T}{}^*_{D_S}(S^* \times S) = \overset{\circ}{T}{}^*_{\overset{\circ}{D}_S}(S^* \times S)$$

and

$$\mathrm{R}p_{2*}\mathrm{R}\Gamma_{D_S} p_1^{-1}\mathcal{G} = \mathrm{R}p_{2*}\mathrm{R}\underline{\mathrm{Hom}}(\underline{A}_{\overset{\circ}{D}_S}, p_1^{-1}\mathcal{G}).$$

In the following part, we show that the conditions in Proposition 3.4.4 are satisfied. Let (z) be a coordinate system of Z, (z,ζ) its associated coordinate system of T^*Z, (z,x) a homogeneous coordinate system of S, $(z,x;\zeta,\xi)$ its associated homogeneous coordinate system of T^*S, (z,y) the dual of (z,x) and $(z,y;\zeta,\eta)$ the dual of $(z,x;\zeta,\xi)$. We have

$$(z,x;\zeta,\xi) \in \overset{\circ}{T}{}^*S \iff |x| = 1, |\xi| \neq 0, < x, \xi >= 0$$
$$(z,y;\zeta,\eta) \in \overset{\circ}{T}{}^*S^* \iff |y| = 1, |\eta| \neq 0, < y, \eta >= 0$$

and

$$(z_1,z_2,x,y;\zeta_1,\zeta_2,\xi,\eta) \in \overset{\circ}{T}{}^*_{\overset{\circ}{D}_S}(S^* \times S) = \Lambda_K$$
$$\iff \begin{cases} z_1 = z_2, |x| = |y| = 1, \zeta_1 + \zeta_2 = 0, \\ \exists t \in \mathbf{R} \setminus \{0\} \quad \text{s.t.} \quad \xi = ty, \eta = tx. \end{cases}$$

Since Claim 2 is a local statement, we may take a neighborhood Ω_S of p_S in $\overset{\circ}{T}{}^*S$ and may restrict Λ_S to Ω_S, Λ_S^* to $\Phi_S(\Omega_S)$ and Λ_K to $\Phi_S(\Omega_S)^a \times \Omega_S$. From now on, we work in the situation under this restriction. Thus we have the equivalence

$$(z_1,z_2,x,y;\zeta_1,\zeta_2,\xi,\eta) \in \Lambda_K$$

$$\iff \begin{cases} z_1 = z_2, |x| = |y| = 1, \zeta_1 + \zeta_2 = 0, \\ \exists t \in \mathbf{R}^+ \quad \text{s.t.} \quad \xi = ty,\ \eta = tx. \end{cases}$$

Then

$$r_1|_{\Lambda_K} : \Lambda_K \longrightarrow \Omega_S$$

is a diffeomorphism. In fact

$$r_1|_{\Lambda_K}((z, z, x, y; \zeta, -\zeta, ty, tx)) = (z, x; \zeta, ty)$$

and

$$(r_1|_{\Lambda_K})^{-1}((z, x; \zeta, \xi)) = (z, z, x, \frac{\xi}{|\xi|}; \zeta, -\zeta, \xi, |\xi| x).$$

The map

$$r_2^a|_{\Lambda_K} : \Lambda_K \longrightarrow \Phi_S(\Omega_S)$$

is also diffeomorphic. Let $p_{SS} = (r_1|_{\Lambda_K})^{-1}(p_S)$. Then $r_1|_{\Lambda_K}$ is transversal to Λ_S at p_{SS} since $r_1|_{\Lambda_K}$ and $r_2^a|_{\Lambda_K}$ are diffeomorphic. Moreover we have

$$r_2^a|_{\Lambda_K}((r_1|_{\Lambda_K})^{-1}(\Lambda_S)) = \Phi_S(\Lambda_S)$$

and $\underline{A}_{\overset{\circ}{D}_S}$ is pure with shift $\frac{1}{2}\operatorname{codim}_{S^* \times S} \overset{\circ}{D}_S$ along Λ_K. It is clear that r_2 is proper on $\operatorname{supp}(\underline{A}_{\overset{\circ}{D}_S}) \cap r_1^{-1}(\operatorname{supp}(\mathcal{G}))$. Thus we can apply Proposition 3.4.4 to this situation. Then $\mathrm{R}p_{2*}\mathrm{R}\underline{\mathrm{Hom}}(\underline{A}_{\overset{\circ}{D}_S}, p_1^{-1}\mathcal{G})$ is pure with shift d'' along Λ_S^* at p_S^* where

$$\begin{aligned} d'' &= d - \frac{1}{2}\operatorname{codim}_{S^* \times S} \overset{\circ}{D}_S - \frac{1}{2}\dim S + \frac{1}{2}\tau \\ &= d - \frac{1}{2}n - \dim Z + \frac{1}{2}\tau \end{aligned}$$

and

$$\tau = \tau(\lambda_0(p_S), \Phi_S^{-1}(\lambda_0(p_S^*)), \lambda_{\Lambda_S}(p_S)).$$

By Proposition 3.2.1 (ii), we have

$$\tau = \tau(\lambda_0(p), \Phi_E^{-1}(\lambda_0(p^*)), \lambda_\Lambda(p)).$$

Now, from Lemma 4.1.4, we deduce

$$\begin{aligned}\mathcal{G}^{\wedge S}[-\dim Z] &= \mathrm{R}p_{2*}\,\mathrm{R}\Gamma_{D_S} p_1^{-1}\mathcal{G} \\ &= \mathrm{R}p_{2*}\,\mathrm{R}\underline{\mathrm{Hom}}(\underline{A}_{\overset{\circ}{D_S}}, p_1^{-1}\mathcal{G}).\end{aligned}$$

Finally we have Claim 2. ∎

From Claim 1 and Claim 2 we get the following claim.

$$\begin{cases} (\mathrm{R}f_{1*}g_1^{-1}\mathcal{F})^{\wedge S} \quad \text{is pure of type } M \text{ with shift} \\ d' = d - \dfrac{n}{2} + \dfrac{1}{2}\tau(\lambda_0(p), \Phi_E^{-1}(\lambda_0(p^*)), \lambda_\Lambda(p)) \\ \text{along } \Lambda_S^* \text{ at } p_S^*. \end{cases}$$

From Lemma 4.1.2 we have

$$(g_1^{-1}\mathcal{F})^{\wedge\circ} = f_2^{-1}(\mathrm{R}f_{1*}g_1^{-1}\mathcal{F})^{\wedge S}.$$

Since f_2 is a projection, we can apply Theorem 3.4.3. Then we find out $(g_1^{-1}\mathcal{F})^{\wedge\circ}$ is pure of type M with shift d' along Λ^* at p^*. Considering the last statement, (4.1.1.1) and the microlocal uniqueness of pure sheaves (see 3.5), we find $g_2^{-1}(\mathcal{F}^\wedge)$ is pure of type M with shift d' along Λ^* at p^*. Now $\mathcal{F}^\wedge$ coincides with $g_2^{-1}(\mathcal{F}^\wedge)$ in a neighborhood of $\pi(p^*)$ because $p^* \in \overset{\circ}{T}{}^*\overset{\circ}{E}{}^*$. Therefore $\mathcal{F}^\wedge$ is pure of type M with shift d' along Λ^* at p^*. Thus the proof of the theorem is finished in the case of $p \in \overset{\circ}{T}{}^*\overset{\circ}{E}$.

2. Proof in the case that $p \in T^*E \setminus \overset{\circ}{T}{}^*\overset{\circ}{E}$.

Define $\mathcal{F}' \in \mathrm{Ob}(D^+(E \underset{Z}{\times} \mathbf{R}^2))$ as follows.

$$\mathcal{F}' = \mathcal{F} \boxtimes \underline{Z}_{\{0\}} \boxtimes \underline{Z}_{\mathbf{R}}$$

Remark

$$\mathrm{SS}(\mathcal{F}') = \mathrm{SS}(\mathcal{F}) \times T^*_{\{0\}}\mathbf{R} \times T^*_{\mathbf{R}}\mathbf{R}.$$

Then, from Proposition 3.4.1 (ii), it follows that $\mathcal{F}'$ is pure of type M with shift $d + \frac{1}{2}$ along $\Lambda \times T^*_{\{0\}}\mathbf{R} \times T^*_{\mathbf{R}}\mathbf{R}$ at $(p, (0;1), (1;0))$. Now we can apply the above result. Then we have $\mathcal{F}'^\wedge$ is pure with shift

$$d'' = d + \frac{1}{2} - \frac{n+2}{2} + \frac{1}{2}\tau_{E\underset{Z}{\times}\mathbf{R}^2}(\lambda_0(p'), \Phi^{-1}_{E\underset{Z}{\times}\mathbf{R}^2}(\lambda_0(p'^*)), \lambda_{\Lambda'}(p'))$$

along $\Lambda^* \times T^*_{\mathbf{R}}\mathbf{R} \times T^*_{\{0\}}\mathbf{R}$ at $(p^*, (1;0), (0;1))$ where

$$\begin{aligned} p' &= (p, (0;1), (1,0)), \\ \Lambda' &= \Lambda \times T^*_{\{0\}}\mathbf{R} \times T^*_{\mathbf{R}}\mathbf{R}, \end{aligned}$$

and

$$p'^{*} = \Phi_{\underset{z}{E\times\mathbf{R}^2}}(p').$$

By Proposition 3.2.1 (ii), we have

$$\begin{aligned}&\tau_{\underset{z}{E\times\mathbf{R}^2}}(\lambda_0(p'), \Phi^{-1}_{\underset{z}{E\times\mathbf{R}^2}}(\lambda_0(p'^{*})), \lambda_{\Lambda'}(p'))\\ =&\tau_E(\lambda_0(p), \Phi_E^{-1}(\lambda_0(p^*)), \lambda_\Lambda(p)).\end{aligned}$$

Then

$$d'' = d + \frac{1}{2} - \frac{n+2}{2} + \frac{1}{2}\tau_E(\lambda_0(p), \Phi_E^{-1}(\lambda_0(p^*)), \lambda_\Lambda(p)).$$

Since $\mathcal{F}'^{\wedge} = \mathcal{F}^{\wedge} \boxtimes \underline{Z}_{\mathbf{R}} \boxtimes \underline{Z}_{\{0\}}[-1]$, it follows from Proposition 3.4.1 (ii) that $\mathcal{F}^{\wedge}$ is pure with shift

$$d - \frac{n}{2} + \frac{1}{2}\tau(\lambda_0(p), \Phi_E^{-1}(\lambda_0(p^*)), \lambda_\Lambda(p))$$

along Λ^* at p^*. Thus the proof of Theorem 4.1.1 is completed. ∎

§5. Application.

As a corollary of Theorem 4.1.1, we prove that the Fourier-Sato transformation of perverse sheaves with n-shifts are also perverse.

5.1 Perverse Sheaves.

We do not recall here the definition of stratification, constructible sheaves and perverse sheaves. Refer to [K-S] and [G-M] for these definitions. We denote by $D^b_{\mathbf{C}-c}(X)$ the subcategory of $D^b(X)$ consisting of C-constructible complexes. For a complex manifold X, we denote by $X^{\mathbf{R}}$, the real underlying manifold of X.

5.2 Perverse Sheaves and Pure Sheaves.

First we have the following theorem from [K-S] which describes the relation between perverse sheaves and pure sheaves.

Theorem 5.2.1 ([K-S, Theorem 9.5.2]). *Let X be a complex manifold, $\mathcal{F}$ be an object of $D^b_{\mathbf{C}-c}(X)$ and $\Lambda = \mathrm{SS}(\mathcal{F})$. Then the following conditions are equivalent.*

(a) $\mathcal{F}$ is a perverse sheaf.

(b) At any point of the non-singular locus Λ_{reg} of Λ, $\mathcal{F}$ is pure with shift 0.

5.3 Fourier-Sato transformation of perverse sheaves.

Let X be a complex vector budle with finite fibre dimension n. When we apply the Fourier-Sato transformation to objects of $D^+(X)$, we regard X as $X^{\mathbf{R}}$ and the objects as those of $D^+(X^{\mathbf{R}})$.

THEOREM 5.3.1. *For an arbitrary object $\mathcal{F}$ of $D^b_{\mathbb{C}-c}(X)$, $\mathcal{F}$ is perverse if and only if $\mathcal{F}^\wedge[n]$ is perverse.*

This proposition was proved by Kashiwara-Schapira [**K-S2**] and analogous propositions in the algebraic category and in the category of F_q were proved by J. L. Brylinsky in [**B**, corollaire 7.23 and corollaire 9.11]. We give a different proof by use of Theorem 4.1.1.

PROOF: Assume $\mathcal{F}$ is perverse. Set $\Lambda = \mathrm{SS}(\mathcal{F})$. Let p be a point of Λ_{reg}. By Theorem 5.2.1, $\mathcal{F}$ is pure with shift 0 along Λ at p. Since $\mathcal{F}$ is perverse, we can regard Λ as $T^*_Y X$ where Y is a smooth submanifold of X ([**K-S**]). Thus we deduce from Theorem 4.1.1 that $\mathcal{F}^\wedge$ is pure with shift d along $\Lambda^* = \Phi_{X^\blacksquare}(T^*_Y X)$ at $p^* = \Phi_{X^\blacksquare}(p)$ where

$$d = -\frac{2n}{2} + \frac{1}{2}\tau(\lambda_0(p), \Phi^{-1}_{X^\blacksquare}(\lambda_0(p^*)), \lambda_{T^*_Y X}(p)).$$

By Proposition 3.2.1 (ii), we have

$$\tau(\lambda_0(p), \Phi^{-1}_{X^\blacksquare}(\lambda_0(p^*)), \lambda_{T^*_Y X}(p)) = 0.$$

From Theorem 5.2.1 again, it follows that $\mathcal{F}^\wedge[n]$ is perverse. vice versa. ∎

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